

HILBERT FUNCTION SPACES AND THE NEVANLINNA-PICK PROBLEM ON THE POLYDISC II

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ABSTRACT. In [19], a geometric procedure for constructing a Nevanlinna-Pick problem on \mathbb{D}^n with a specified set of uniqueness was established. In this sequel we conjecture a necessary and a sufficient condition for a Nevanlinna-Pick problem on \mathbb{D}^2 to have a unique solution. We use the results of [19] and Bezout's theorem to establish three special cases of this conjecture.

1. OVERVIEW

The *Schur class* of the n -disc, $\mathcal{S}(\mathbb{D}^n)$, is the set of analytic functions mapping \mathbb{D}^n to $\overline{\mathbb{D}}$, i.e. satisfying $\|F\|_\infty = \sup_{z \in \mathbb{D}^n} |f(z)| \leq 1$. The Nevanlinna-Pick problem on \mathbb{D}^n is to determine, given distinct nodes $\lambda_1, \dots, \lambda_N \in \mathbb{D}^n$ and target points $\omega_1, \dots, \omega_N \in \mathbb{D}$, whether there exists a function $F \in \mathcal{S}(\mathbb{D}^n)$ that satisfies $F(\lambda_i) = \omega_i$ for each i . We are primarily interested in the following question.

Question 1.1. *What are necessary and sufficient conditions for a Nevanlinna-Pick problem on \mathbb{D}^2 to have a unique solution?*

Various authors have studied the uniqueness properties of the Nevanlinna-Pick problem: in [6] Ball and Trent show how to parameterize the set of all solutions associated to a given problem on \mathbb{D}^2 ; in [2] Agler and McCarthy classify those 2 and 3 point problems on \mathbb{D}^2 that have a unique solution; in [15] Knese gives sufficient conditions for a 4 point problem on \mathbb{D}^2 to have a unique solution; in [13] Guo, Huang and Wang give sufficient conditions for a 3 point Pick problem on \mathbb{D}^3 to have a unique solution; in [18], the present author gives sufficient conditions for a Nevanlinna-Pick problem on \mathbb{D}^n to have a unique solution; in [19], the present author gives a geometric procedure for constructing a Nevanlinna-Pick problems on \mathbb{D}^n with a specified set of uniqueness.

In this work we introduce the notion of a *strong Pick set* and a question closely related to Question 1.1. To state them we recall that a rational function $f \in \mathcal{S}(\mathbb{D}^n)$ is called *inner* if $|f| = 1$ almost everywhere on \mathbb{T}^n and that an irreducible algebraic variety $V \subset \mathbb{C}^n$ is called *inner* if it meets \mathbb{D}^n and exits \mathbb{D}^n through the n -torus, i.e. $V \cap \mathbb{D}^n \neq \emptyset$ and $V \cap \partial(\mathbb{D}^n) \subset \mathbb{T}^n$.

Definition 1.2. *Given a function $f \in \mathcal{S}(\mathbb{D}^n)$ and an inner variety $V \subset \mathbb{C}^n$, we say that V is a strong Pick set for f , if each $h \in \mathcal{S}(\mathbb{D}^n)$ that equals f on $V \cap \mathbb{D}^n$ equals f on \mathbb{D}^n , i.e. if $h|_V = f|_V$, then $h = f$.*

Question 1.3. *Given a rational inner function $f \in \mathcal{S}(\mathbb{D}^2)$ and an inner variety V , what are necessary and sufficient conditions for V to be a strong Pick set for f ?*

The degree of a rational inner function f on \mathbb{D}^n , denoted $\deg(f)$, is the degree of the numerator of f in an irreducible representation. The degree of a rational inner function f on \mathbb{D}^n in z_i , denoted $\deg_i(f)$, is the degree of such a numerator in z_i . On \mathbb{D} , the answer to Question 1.3 is given by the following corollary of Pick's 1916 results.

Corollary 1.4. *For a polynomial p with zeros given by distinct points in \mathbb{D} and a rational inner function f , $V = Z_p$ is a strong Pick set for f if and only if $\deg(f) < \deg(p)$.*

We state Pick's original result and derive Corollary 1.4 in Section 6. Our conjecture is that Corollary 1.4 generalizes to \mathbb{D}^2 .

Conjecture 1.5. *Fix a rational inner function f on \mathbb{D}^2 and an irreducible inner variety $V = Z_p$.*

If $\deg_i(f) < \deg_i(p)$ for $i = 1, 2$, then V is a strong Pick set for f .

If $\deg_i(f) \geq \deg_i(p)$ for $i = 1, 2$, then V is not a strong Pick set for f .

The main results of this paper are several special cases of Conjecture 1.5. We do not address the mixed case $\deg_1(f) < \deg_1(p)$ and $\deg_2(f) \geq \deg_2(p)$ since there exist such examples where V is and is not a strong Pick set for f . We also mention that the following partial case of this conjecture was established in [18].

Theorem 1.6. *(Scheinker, [18]) Fix positive integers n and N . There exists a 1-dimensional inner variety $V \subset \mathbb{C}^n$ with the following property. V is a strong Pick set for each rational inner function f on \mathbb{D}^n that satisfies $\deg(f) < N$.*

This paper is organized as follows. In the Section 2 we give some background and establish the relationship between Question 1.1 and Question 1.3. In the Section 3 we state our main results. In Sections 4, 5 and 6 we prove our main results.

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2. BACKGROUND

A Nevanlinna-Pick problem on \mathbb{D}^n is called *extremal* if a solution f satisfying $\|f\|_\infty = 1$ exists and no solution h satisfying $\|h\|_\infty < 1$ exists. If a problem is not extremal, then it does not have a unique solution. Indeed, if there exists a solution f with $\|f\|_\infty < 1$, then for any polynomial p vanishing on the nodes and any $g \in \mathcal{S}(\mathbb{D}^n)$ of sufficiently small norm, $f + pg$ is a solution. On \mathbb{D} , the condition of being extremal is sufficient for a problem to have a unique solution. The following example shows that on \mathbb{D}^2 , unlike on \mathbb{D} , a Nevanlinna-Pick problem may be extremal and fail to have a unique solution.

Example 2.1 On \mathbb{D}^2 , the problem with data $(0, 0), (\frac{1}{2}, \frac{1}{2})$ and $0, \frac{1}{2}$ is extremal and fails to have a unique solution. Let $V = Z_p$ where $p = z - w$. If f is a solution, then $f_d(z) = f|_V = f(z, z)$ is in $\mathcal{S}(\mathbb{D})$, satisfies $f_d(0) = 0$, $f_d(1/2) = 1/2$ and the classical Schwarz lemma implies that $f_d(z) = z$. Thus, all solutions to the problem agree on $V \cap \mathbb{D}^2$ and the problem is extremal since $\|f\|_\infty \geq \|f_d\|_\infty = 1$. The solution is not unique since each coordinate function solves.

Example 2.1 is representative of those extremal Nevanlinna-Pick problems on \mathbb{D}^2 that fail to have a unique solution, in a sense made precise by the following three theorems.

Theorem 2.2. (Agler, [1]): *If a Nevanlinna-Pick problem on \mathbb{D}^2 has a solution, then it has a solution that is a rational inner function.*

Theorem 2.3. (Agler and McCarthy, [4]): *Given an extremal Nevanlinna-Pick problem on \mathbb{D}^2 , there exists an inner variety V with the property that all solutions agree on $V \cap \mathbb{D}^2$.*

Theorem 2.4. (Scheinker, [19]): *Given a rational inner function f and an inner variety $V = Z_p$ there exists a Nevanlinna-Pick problem on \mathbb{D}^2 with nodes lying on V such that each solution to the problem equals f on $V \cap \mathbb{D}^2$.*

These theorems allow us to demonstrate the relationship between Question 1.1 and Question 1.3. Suppose that the problem with data $\lambda_1, \dots, \lambda_N$ and $\omega_1, \dots, \omega_N$ has a unique solution f . Theorem 2.2 implies that f is a rational inner function and the proof of Theorem 2.3 guarantees the existence of an inner variety V containing the nodes $\lambda_1, \dots, \lambda_N$. If $g \in \mathcal{S}(\mathbb{D}^2)$ equals f on V , then f is another solution to the problem and $g = f$ on \mathbb{D}^2 . Thus, V is a strong Pick set for f . Conversely, suppose that f is a rational inner function and that V is a strong Pick set for f . Theorem 2.4 guarantees the existence of with nodes lying on $V \cap \mathbb{D}^2$ with the property that f is a solution and that all solutions agree on $V \cap \mathbb{D}^2$. If g is another solution to the problem, then g equals f on V which implies that $g = f$. Thus, the problem has a unique solution.

3. STATEMENT OF MAIN RESULTS

Our first main result allows us to establish several cases of Conjecture 1.5. It is stated using the inner product and the norm of the Hardy space of \mathbb{D}^2 . The Hardy space of \mathbb{D}^2 , denoted H^2 , is the Hilbert space of analytic functions on \mathbb{D}^2 with square summable Taylor coefficients at $(0, 0)$ and norm and inner product given by the following equivalent formulas (we recommend [3] for a concise presentation of the pertinent facts about H^2). For $f = \sum_0^\infty a_n z^n$ and $g = \sum_0^\infty b_n z^n$ in H^2 ,

$$\langle f, g \rangle = \sum_0^\infty a_n \overline{b_n} = \int_{\mathbb{T}^2} f \overline{g} dm \text{ and } \|f\|_2^2 = \sum_0^\infty |a_n|^2 = \int_{\mathbb{T}^2} |f|^2 dm.$$

Theorem 3.1. *Fix a rational inner function f and an inner variety $V = Z_p$. If for each function g analytic on \mathbb{D}^2 such that pg is bounded the inequality*

$$2\operatorname{Re}(\langle f, pg \rangle) < \|pg\|_2^2$$

holds, then V is a strong Pick set for f .

Theorem 3.1 is somewhat surprising since the norm of the Hardy space on \mathbb{D}^2 is not equivalent to the infinity norm on \mathbb{D}^2 in which the Nevanlinna-Pick problem is stated. The usefulness of Theorem 3.1 is, of course, contingent on the difficulty of showing that the hypothesis holds. To demonstrate the applicability of Theorem 3.1 we mention that the following result, of independent interest, is an almost immediate corollary.

Theorem 3.2. Fix $f = z_1^{d_1} z_2^{d_2}$, fix an inner variety $V = Z_p$ and write p as a sum of monomials, $p = m_1 + \dots + m_k$. If for each m_k , $\deg_i(f) < \deg_i(m_k)$ for $i = 1$ or $i = 2$, then V is a strong Pick set for f .

To examine the implications of Theorem 3.2, one could try proving directly the special case when $f = z_1 z_2$ and $p = z_1^2 - z_2^2$, i.e. if $g \in \mathcal{S}(\mathbb{D}^2)$ equals f on the zero set of p , then $g = z_1 z_2$. The present author is unaware of a simple proof of this seemingly simple result. We prove Theorem 3.1 and derive Theorem 3.2 as a corollary in Section 4.

Our second main result establishes Conjecture 1.5 for *regular* rational inner functions, rational inner functions that are continuous on a neighborhood of \mathbb{D}^2 .

Theorem 3.3. Fix a regular rational inner f and an irreducible inner variety $V = Z_p$. If $\deg_i(f) < \deg_i(p)$ for $i = 1, 2$ and h is a regular rational inner function that satisfies $h|_V = f|_V$, then $h = f$.
If $\deg_i(f) \geq \deg_i(p)$ for $i = 1, 2$, then V is not a strong Pick set for f .

We prove Theorem 3.3 in Section 5 using Bezout's Theorem.

Our last main result is a complete classification of extremal minimal Nevanlinna-Pick problems on \mathbb{D}^2 that have a solution of one variable only. One may expect the study of such a problem to reduce trivially to the study of a problem on \mathbb{D} . However, this is not the case since, as becomes evident from the proof of Theorem 3.4, there may exist a non-trivial geometric relationship between the first and second coordinates of the nodes.

Theorem 3.4. Fix an extremal minimal Nevanlinna-Pick problem that has a solution f , a function of z_1 only. There exists an Blaschke product of one variable $m(\lambda)$ and inner variety $V = Z_p$ such that $V \cap \mathbb{D}^2 = \{(\lambda, m(\lambda)) : \lambda \in \mathbb{D}\}$ contains the nodes of the problem and all solutions to the problem agree on V . Furthermore, one of the following holds.
If $\deg_1(f) < \deg_1(p)$, then V is a strong Pick set for f and f is the unique solution.
If $\deg_1(f) \geq \deg_1(p)$, then V is not a strong Pick set for f and f is not the unique solution.

We prove Theorem 3.4 in Section 6 by generalizing an argument from [3].

4. PROOF OF THEOREM 3.1

In this section we prove Theorem 3.1 and derive Theorem 3.2 as a corollary.

PROOF OF THEOREM 3.1: Fix a rational inner function f , an inner variety $V = Z_p$ and suppose that for each function g analytic on \mathbb{D}^2 such that $f - pg$ is bounded the following inequality holds:

$$2\operatorname{Re}(\langle f, pg \rangle) < \|pg\|_2^2.$$

Suppose, towards a contradiction, that there exists $r \in \mathcal{S}(\mathbb{D}^2)$ such that $r|_V = f|_V$ and $r \neq f$. We first show that there exists a rational inner function h that satisfies $h|_V = f|_V$ and $h \neq f$. By Theorem 2.4 there exists a Nevanlinna-Pick problem with nodes $\lambda_1, \dots, \lambda_N \in V \cap \mathbb{D}^2$ and target values $f(\lambda_1), \dots, f(\lambda_N)$ such that all solutions agree on $V \cap \mathbb{D}^2$. Since $r \neq f$, there exists a $\lambda_{N+1} \in \mathbb{D}^2$ such that $f(\lambda_{N+1}) \neq r(\lambda_{N+1})$. Consider the Nevanlinna-Pick problem with nodes $\lambda_1, \dots, \lambda_N, \lambda_{N+1}$ and target values $r(\lambda_1), \dots, r(\lambda_{N+1})$. The problem is solvable since r is a solution and Theorem 2.2 implies that there exists a

rational inner solution h . But notice, since h is also a solution to the problem with data $\lambda_1, \dots, \lambda_N$ and $f(\lambda_1), \dots, f(\lambda_N)$, h equals f on $V \cap \mathbb{D}^2$.

Write $f - h = pg$ where g is analytic on \mathbb{D}^2 and notice that pg is bounded on \mathbb{D}^2 since it is the difference of two bounded functions.

$$(4.1) \quad 1 = \|h\|_\infty^2$$

$$(4.2) \quad = \|f - pg\|_\infty^2$$

$$(4.3) \quad = \int_{\mathbb{T}^2} |f - pg|^2 dm$$

$$(4.4) \quad = \|f - pg\|_2^2$$

$$(4.5) \quad = \|f\|_2^2 - 2\operatorname{Re} \langle f, pg \rangle + \|pg\|_2^2$$

$$(4.6) \quad = 1 - 2\operatorname{Re} \langle f, pg \rangle + \|pg\|_2^2$$

Thus, $2\operatorname{Re} \langle f, pg \rangle = \|pg\|_2^2$ which contradicts our assumption. The equality of 4.2 and 4.3 follows from the fact that $h = f - pg$ is inner, i.e. has modulus equal to 1 almost everywhere on \mathbb{T}^2 . \square

We now prove Theorem 3.2. Fix a rational inner function $f = z_1^{d_1} z_2^{d_2}$ and an inner variety $V = Z_p$. Write p as the sum of monomials $p = m_1 + \dots + m_k$ ordered so that for $j = 1, \dots, l$, $\deg_1(f) < \deg_1(m_j)$ and for $j = l + 1, \dots, k$, $\deg_2(f) < \deg_2(m_j)$.

$$(4.7) \quad \langle f, pg \rangle = \langle z_1^{d_1} z_2^{d_2}, (m_1 + \dots + m_k)g \rangle$$

$$(4.8) \quad = \sum_{j=1}^k \langle z_1^{d_1} z_2^{d_2}, m_j g \rangle$$

$$(4.9) \quad = \sum_{j=1}^k \int_{\mathbb{T}^2} z_1^{d_1} z_2^{d_2} \overline{m_j g} dm$$

$$(4.10) \quad = \sum_{j=1}^l \int_{\mathbb{T}^2} z_2^{d_2} \overline{z_1^{-d_1} m_j g} dm + \sum_{j=l+1}^k \int_{\mathbb{T}^2} z_1^{d_1} \overline{z_2^{-d_2} m_j g} dm$$

$$(4.11) \quad = \sum_{j=1}^l \langle z_2^{d_2}, z_1^{-d_1} m_j g \rangle + \sum_{j=l+1}^k \langle z_1^{d_1}, z_2^{-d_2} m_j g \rangle$$

$$(4.12) \quad = 0.$$

Thus, $\langle f, pg \rangle = 0$ and the conclusion follows from Theorem 3.1. The equality of 4.9 and 4.10 follows from the fact that on \mathbb{T}^2 one has $z_i^a = \overline{z_i^{-a}}$. The equality $\langle z_2^{d_2}, z_1^{-d_1} m_j g \rangle = 0$ for $j = 1, \dots, l$ follows from noticing that the Taylor coefficients of the $z_2^{d_2}$ term in the Taylor series of $z_1^{-d_1} m_j g$ is zero and the equality $\langle z_1^{d_1}, z_2^{-d_2} m_j g \rangle = 0$ follows by an analogous consideration.

5. PROOF OF THEOREM 3.3

The second part of Theorem 3.3 is an immediate consequence of the following result.

Theorem 5.1. (*Scheinker [19]*) *Let f be a regular rational inner function and $V = Z_p$ an inner variety. If $\deg_i(f) \geq \deg_i(p)$ for $i = 1, 2$, then there exists a rational inner function g that equals f on V and does not equal f on \mathbb{D}^2 .*

To prove the first of part Theorem 3.3 consider a regular rational inner function f with $\deg_i(f) = d_i$ and an inner variety $V = Z_p$ with $\deg_i(p) = n_i$ such that $d_1 < n_1$ and $d_2 < n_2$. Let g be a regular rational inner function with $\deg_i(g) = e_i$ such that $g = f$ on V . Assume, towards a contradiction, that $g \neq f$ on \mathbb{D}^2 .

The way we have set things up, Theorem 2.8 of [4] implies that the number of zeros of f on V is $d_1 n_2 + d_2 n_1$ and the number of zeros of g on V is $e_1 n_2 + e_2 n_1$. The assumption that $g = f$ on V implies that $d_1 n_2 + d_2 n_1 = e_1 n_2 + e_2 n_1$. If we can show that f and g have at most $d_1 e_2 + d_2 e_1$ common zeros in \mathbb{C}^2 , then we will have the following inequality

$$e_1 n_2 + e_2 n_1 = |Z_f \cap Z_g \cap Z_p| \leq |Z_f \cap Z_g| = e_1 d_2 + e_2 d_1,$$

which contradicts the assumption that $d_1 < n_1$ and $d_2 < n_2$.

Thus, the proof of Theorem 3.3 will be complete once we establish the following theorem.

Theorem 5.2. *Let f and g be rational inner functions of degree (d_1, d_2) and (e_1, e_2) . The number of common zeros of f and g counted with multiplicity is less than or equal to $d_1 e_2 + d_2 e_1$. That is, $|Z_f \cap Z_g| \leq d_1 e_2 + d_2 e_1$.*

For the reader's convenience we now recall the definitions and results used to state Bezout's Theorem and prove Theorem 5.2. Rather than discuss the notion of a general algebraic variety in \mathbb{C}^n given as the intersection of the zero sets of several polynomials, we specialize the presentation of [12] to emphasize the notion of a plane algebraic curve, an algebraic variety in \mathbb{C}^2 given as the zero set of a single polynomial. To simplify notation and keep with the notation of [12] and we use the variables (x, y) instead of (z_1, z_2) .

Definition 5.3. (I.8.1 [12]) *An affine algebraic curve is a subset of \mathbb{C}^2 defined by*

$$V = \{(x, y) \in \mathbb{C}^2 : p(x, y) = 0\},$$

where p is a polynomial. The degree of V is the degree of p . We write $V = Z_p$.

Definition 5.4. (I.8.2 [12]) *A projective algebraic curve is a subset of $\mathbb{P}^2\mathbb{C}$ defined by*

$$V = \{(x, y, z) \in \mathbb{P}^2\mathbb{C} : P(x, y, z) = 0\},$$

where P is a homogeneous polynomial. The degree of V is the degree of P . We write $V = Z_P$.

We use the natural embedding of \mathbb{C}^2 into $\mathbb{P}^2\mathbb{C}$ that identifies the points $(x, y) \in \mathbb{C}^2$ and $(x, y, 1) \in \mathbb{P}^2\mathbb{C}$. We will abuse notation and write $(x, y) \in \mathbb{C}^2 \subset \mathbb{P}^2\mathbb{C}$. This identification allows us to identify an affine algebraic curve $V = Z_p$ of degree n with the projective algebraic curve $V = Z_P$ of degree n as follows. Given $p(x, y)$ of degree n , let $P(x, y, z) = z^n p(\frac{x}{z}, \frac{y}{z})$ and given $P(x, y, z)$, let $p(x, y) = P(x, y, 1)$.

Lemma 5.5. (II.5.1 [12]) *Suppose V is a projective algebraic curve with $\lambda \in V$. There exists a coordinate system such that*

$$\lambda = (0, 0) \in \mathbb{C}^2 \subset \mathbb{P}^2 C,$$

and such that the affine equation of V , given by $p(x, y) = 0$, satisfies

$$p(x, y) = y^k + a_1(x)y^{k-1} + \dots + a_k(x),$$

where $a_j(x)$ is a polynomial of degree less than or equal to j or $a_j(x) = 0$.

Definition 5.6. (II.7.3 [12]) *Suppose the affine algebraic curves $V = Z_p$ and $W = Z_q$ intersect at the point λ . After a suitable change of coordinates, we may assume that $\lambda = (0, 0)$ and that the conclusion of Lemma 5.5 holds. If p is locally irreducible in a neighborhood of $(0, 0)$, then there exists a local normalization of V at $(0, 0)$ given by $g : \mathbb{D} \rightarrow \mathbb{D}^2$ with*

$$g(t) = (t^k, y_v(t^k)),$$

and we define the intersection number of V and W at $\lambda = (0, 0)$ as the multiplicity of the zero of the one variable analytic function $h(g(t))$ at $t = 0$. In the general case, suppose that in a neighborhood of $\lambda = (0, 0)$, p factors as $p = p_1^{m_1} \cdot \dots \cdot p_l^{m_l}$ where each p_j is locally irreducible in a neighborhood of $\lambda = (0, 0)$. Let $V_j = Z_{p_j}$ and define the intersection number of V and W at $\lambda = (0, 0)$ as

$$(V \cdot W)_\lambda = \sum_{j=1}^l m_j (V_j \cdot W)_\lambda.$$

Definition 5.7. (II.7.4 [12]) *The intersection number of two projective algebraic curves V and W in $\mathbb{P}^2 \mathbb{C}$ is*

$$(V \cdot W) = \sum_{\lambda \in V \cap W} (V \cdot W)_\lambda.$$

Theorem 5.8. (II.7.5 Bezout)

Suppose two projective algebraic curves $V = Z_P$ and $W = Z_Q$ have no common curve components (i.e. the polynomials P and Q have no common factor). Then

$$(V \cdot W) = \deg(V) \cdot \deg(W) = \deg(P) \cdot \deg(Q).$$

Finally, consider two rational inner functions as the ratios of irreducible polynomials $f_1 = \frac{q}{q_d}$, $f_2 = \frac{r}{r_d}$ and an inner variety $V = Z_p$ with p irreducible. Let Q, S and P denote the projective polynomials associated to q, s and p . Define the *number of common zeros of f_1 and f_2* as the sum of the intersection numbers of the projective curves Z_Q and Z_S at points $\lambda \in \mathbb{C}^2$, i.e.

$$|Z_{f_1} \cap Z_{f_2}| = \sum_{\lambda \in (Z_Q \cap Z_S) \cap \mathbb{C}^2} (Z_Q \cap Z_S)_\lambda.$$

Define the *number of zeros of f on $V \cap \mathbb{D}^2$* as the intersection numbers of the projective curves Z_Q and Z_P at points in \mathbb{D}^2 , i.e.

$$\deg_V(f) = \sum_{\lambda \in (Z_Q \cap Z_P) \cap \mathbb{D}^2} (Z_Q \cap Z_P)_\lambda.$$

PROOF OF THEOREM 5.2:

Write f_1 and f_2 as the ratios of two polynomials relatively prime in $\mathbb{C}[z, w]$.

$$f_1(x, y) = \frac{q(x, y)}{q_d(x, y)} \quad \text{and} \quad f_2(x, y) = \frac{r(x, y)}{r_d(x, y)}$$

If $\overline{a_n}$ and $\overline{b_m}$ are the, necessarily non-zero, constant terms of q_d and p_d , then Rudin's theorem on the structure of rational inner functions [17] implies that p and q have the form

$$q(x, y) = a_n x^{d_1} y^{d_2} + a_{n-1} x^{d_1-1} y^{d_2} \dots + a_0 \quad \text{and} \quad r(x, y) = b_m x^{e_1} y^{e_2} + b_{m-1} x^{e_1-1} y^{e_2} \dots + b_0$$

Letting $n = d_1 + d_2$ and $m = e_1 + e_2$, the projective polynomials corresponding to q and r have the form

$$Q(x, y, z) = z^n q\left(\frac{x}{z}, \frac{y}{z}\right) = a_n x^{d_1} y^{d_2} + a_{n-1} x^{d_1-1} y^{d_2} z \dots + a_0 z^n$$

$$R(x, y, z) = z^m r\left(\frac{x}{z}, \frac{y}{z}\right) = b_m x^{e_1} y^{e_2} + b_{m-1} x^{e_1-1} y^{e_2} z \dots + b_0 z^m$$

Bezout's theorem tells us that the intersection number is

$$\deg(Q) \cdot \deg(R) = (d_1 + d_2)(e_1 + e_2) = d_1 e_1 + d_2 e_2 + d_1 e_2 + d_2 e_1.$$

The intersection number of these polynomials at infinity is at points in $\mathbb{P}^2 \mathbb{C}$ of the form $\{x, y, 0\}$. At these points the polynomials take the form

$$Q(x, y, 0) = a_n x^{d_1} y^{d_2} \quad \text{and} \quad R(x, y, 0) = b_m x^{e_1} y^{e_2}.$$

Breaking these up gives

$$Q(x, y, 0) = a_n x^{d_1} \quad \text{and} \quad R(x, y, 0) = b_m x^{e_1} \quad \text{at points of the form } \{x, 1, 0\}.$$

$$Q(x, y, 0) = a_n y^{d_2} \quad \text{and} \quad R(x, y, 0) = b_m y^{e_2} \quad \text{at points of the form } \{1, y, 0\}.$$

These intersect at $\{0, 1, 0\}$ with multiplicity $d_1 e_1$ and at $\{1, 0, 0\}$ with multiplicity $d_2 e_2$.

Subtracting the $d_1 e_1 + d_2 e_2$ intersections at infinity from the intersection number gives $d_1 e_2 + d_2 e_1$ as an upper bound for the number of intersection points of the form $\{x, y, 1\}$, i.e. in \mathbb{C}^2 . Thus, $|Z_{f_1} \cap Z_{f_2}| \leq d_1 e_2 + d_2 e_1$. \square

6. PROOF OF THEOREM 3.4

In this section we characterize problems that have a solution of one variable only by generalizing an argument from Chapter 12 of [3]. In the remainder of this section we use $\{\lambda_i \rightarrow \omega_i\}_1^N$ to denote the Nevanlinna-Pick problem with data $\lambda_1, \dots, \lambda_N$ and $\omega_1, \dots, \omega_N$. Before proving Theorem 3.4, we recall several definitions and results and prove Corollary 1.4 from the introduction.

Theorem 6.1. (*Pick 1916*) *On \mathbb{D} , the following are equivalent.*

- a. *The problem $\{\lambda_i \rightarrow \omega_i\}_1^N$ is solvable.*
- b. *The Pick matrix $P = \left(\frac{1 - \overline{\omega_i} \omega_j}{1 - \overline{\lambda_i} \lambda_j} \right)$ is positive semi-definite.*
- c. *The problem $\{\lambda_i \rightarrow \omega_i\}_1^N$ has a rational inner solution f with $\deg(f) = \text{rank}(P)$. In this case, the following are equivalent.*

- i. The problem has a unique solution.*
- ii. The problem is extremal.*
- iii. The Pick matrix P is singular.*

PROOF OF COROLLARY 1.4: Fix a rational inner function f on \mathbb{D} , fix $V = Z_p$ where p is a polynomial with distinct zeros $\lambda_1, \dots, \lambda_N \in \mathbb{D}$. Consider the problem $\{\lambda_i \rightarrow f(\lambda_i)\}_1^N$ and the associated Pick matrix,

$$P = \left(\frac{1 - \overline{f(\lambda_i)}f(\lambda_j)}{1 - \bar{\lambda}_i\lambda_j} \right)$$

Parts *c* and *iii* of Theorem 6.1 imply that the problem has a unique solution if and only if $\deg(f) = \text{rank}(P) < N = \deg(p)$. Notice that a function $g \in \mathcal{S}(\mathbb{D})$ is a solution if and only if $g|_V = f|_V$. If $\deg(f) < \deg(p)$, then the problem has a unique solution and each $g \in \mathcal{S}(\mathbb{D})$ that satisfies $g|_V = f|_V$ must equal f . If $\deg(f) \geq \deg(p)$, then the problem fails to have a unique solution and there exists a $g \in \mathcal{S}(\mathbb{D})$ such that $g|_V = f|_V$ and $g \neq f$. \square

Given a problem $\{\lambda_i \rightarrow \omega_i\}_1^N$, write $\lambda_i = (\lambda_i^1, \lambda_i^2)$ and let W , Λ^1 and Λ^2 denote the following N -by- N matrices.

$$W = (1 - \bar{\omega}_i\omega_j)_{i,j=1}^N \quad \Lambda^1 = (1 - \bar{\lambda}_i^1\lambda_j^1)_{i,j=1}^N \quad \Lambda^2 = (1 - \bar{\lambda}_i^2\lambda_j^2)_{i,j=1}^N$$

For a matrix A , write $A \geq 0$ if A is positive semi-definite and $A > 0$ if it is positive definite. Let $W \cdot K = (W_{ij}K_{ij})$ denote the Schur entrywise product of two matrices W and K . A positive definite matrix K is an *admissible kernel* if $\Lambda^1 \cdot K \geq 0$ and $\Lambda^2 \cdot K \geq 0$, and K is *active* if $\det(W \cdot K) = 0$. Finally, if the problem is extremal and no $N-1$ point subproblem $\{\lambda_{i_k} \rightarrow \omega_{i_k}\}_1^{N-1}$ is extremal, then the problem is called *minimal*.

Theorem 6.2. (Agler, [1]) *On \mathbb{D}^2 , the following are equivalent.*

- i. The problem $\{\lambda_i \rightarrow \omega_i\}_1^N$ has a solution.*
- ii. For each admissible kernel K , $W \cdot K \geq 0$.*
- iii. There exists a pair of positive semi-definite matrices (Γ, Δ) such that $W = \Gamma \cdot \Lambda^1 + \Delta \cdot \Lambda^2$.*

Lemma 6.3. (Agler, McCarthy [4]) *If $\{\lambda_i \rightarrow \omega_i\}_1^N$ is an extremal Nevanlinna-Pick problem, then there exists an admissible kernel K that is active. Furthermore, if the problem is minimal and K is an active kernel, then $\text{rank}(K \cdot W) = N - 1$.*

We now prove the lemmas we need to establish Theorem 3.4.

Lemma 6.4. *Fix a problem $\{\lambda_i \rightarrow \omega_i\}_1^N$ on \mathbb{D}^2 and let K denote the Szegő kernel of the Hardy space on \mathbb{D}^2 ,*

$$K_{\lambda_i, \lambda_j} = \frac{1}{(1 - \bar{\lambda}_i z_1)(1 - \bar{\lambda}_j z_2)}.$$

If $W \cdot K$ is singular, then the problem has a unique solution.

PROOF: Theorem 1.6 of [19] implies that the generalized problem $\{\lambda_i \rightarrow \omega_i\}_1^N$ in the multiplier algebra of $H^2(\mathbb{D}^2)$, $\text{Mult}(H^2(\mathbb{D}^2))$, has a unique solution. However, since a multiplier M_f is in the unit ball of $\text{Mult}(H^2(\mathbb{D}^2))$ if and only if $f \in \mathcal{S}(\mathbb{D}^2)$, this implies that the original problem also has a unique solution. \square

Lemma 6.5. *Fix a problem $\{\lambda_i \rightarrow \omega_i\}_1^N$ on \mathbb{D}^2 . If there exists a pair of non-zero positive semi-definite matrices (Γ, Δ) such that $W = \Gamma \cdot \Lambda^1 + \Delta \cdot \Lambda^2$, then there exists a solution f that is a function of both z_1 and z_2 .*

PROOF: In Theorem 6.2, the proof of *iii* implies *i* proceeds by showing that the entry wise equalities of $W = \Gamma \cdot \Lambda^1 + \Delta \cdot \Lambda^2$ extend to all of \mathbb{D}^2 in the following sense. There exists a pair of positive semi-definite functions Γ, Δ on $\mathbb{D}^2 \times \mathbb{D}^2$ such that $\Gamma(\lambda_i, \lambda_j) = \Gamma_{ij}$, $\Delta(\lambda_i, \lambda_j) = \Delta_{ij}$ and a rational inner function $f \in \mathcal{S}(\mathbb{D}^2)$ such that

$$(6.13) \quad \forall(\lambda, \zeta) \in \mathbb{D}^2 \times \mathbb{D}^2 \quad 1 - \overline{f(\lambda)}f(\zeta) = (1 - \overline{\lambda^1}\zeta^1)\Gamma(\lambda, \zeta) + (1 - \overline{\lambda^2}\zeta^2)\Delta(\lambda, \zeta)$$

In [8], Cole and Wermer show that if f is written as the ratio of relatively prime polynomials $\frac{\tilde{r}}{r}$ then the following version of the Agler realization holds with A_i and B_i polynomials.

$$(6.14) \quad \overline{r(\lambda)}r(\zeta) - \overline{\tilde{r}(\lambda)}\tilde{r}(\zeta) = (1 - \overline{\lambda^1}\zeta^1) \sum_1^M \overline{A_i(\lambda)}A_i(\zeta) + (1 - \overline{\lambda^2}\zeta^2) \sum_1^M \overline{B_i(\lambda)}B_i(\zeta)$$

Suppose, towards a contradiction, that f does not depend on z_2 . Then, neither \tilde{p} nor p depends on z_2 . Differentiating both sides of 6.14 with respect to ζ^2 gives

$$(6.15) \quad 0 = (1 - \overline{\lambda^1}\zeta^1) \frac{d}{d\zeta_2} \sum_1^M \overline{A_i(\lambda)}A_i(\zeta) + \frac{d}{d\zeta_2} (1 - \overline{\lambda^2}\zeta^2) \sum_1^M \overline{B_i(\lambda)}B_i(\zeta)$$

Notice that if $\frac{d}{d\zeta_2} \sum_1^M \overline{A_i(\lambda)}A_i(\zeta) \neq 0$, then one can solve for $1 - \overline{\lambda^1}\zeta^1$ as a ratio of polynomials that depend on λ^2 and ζ^2 , a contradiction. Thus, $\frac{d}{d\zeta_2} \sum_1^M \overline{A_i(\lambda)}A_i(\zeta) = 0$ and (6.15) can be written as

$$0 = \frac{d}{d\zeta_2} (1 - \overline{\lambda^2}\zeta^2) \sum_1^M \overline{B_i(\lambda)}B_i(\zeta).$$

This implies that $\sum_1^M \overline{B_i(\lambda)}B_i(\zeta) = 0$, which implies that $\Delta(\lambda, \zeta) = 0$ a contradiction. \square

The following lemma is a slightly modified version of Lemma 12.11 in [3].

Lemma 6.6. *Fix an extremal, minimal problem $\{\lambda_i \rightarrow \omega_i\}_1^N$ on \mathbb{D}^2 . If K is an active kernel and (Γ, Δ) is a pair of positive matrices with $\text{rank}(\Gamma) = N-1$ that satisfy $W = \Gamma \cdot \Lambda^1 + \Delta \cdot \Lambda^2$, then $\text{rank}(K \cdot \Lambda^1) = 1$.*

PROOF: If K is an active kernel, then $K \cdot W$ has rank $N-1$ and annihilates some vector $\vec{\gamma} = (\gamma_1, \dots, \gamma_N)$ and since the problem is minimal, each $\gamma_i \neq 0$.

Since Γ is positive we let $\vec{u}^k = (u_1^k, \dots, u_N^k)^T$ and write Γ as the sum of rank one matrices $\Gamma = \sum_1^{N-1} \vec{u}^k \otimes \vec{u}^k$ where none of \vec{u}^k and \vec{u}^l are collinear for $k \neq l$ and $\vec{u}^k \otimes \vec{u}^k$ denotes the

matrix $\left(\vec{u}^k \otimes \vec{u}^k \right)_{ij} = u_i^k \bar{u}_j^k$. Let the rank of $K \cdot \Lambda^1$ equal P and write $K \cdot \Lambda^1 = \sum_1^P \vec{x}^r \otimes \vec{x}^r$.

Notice that $(K \cdot \Lambda^1 \cdot \Gamma) \vec{\gamma} = 0$, since $0 = W \cdot K \gamma = (K \cdot \Lambda^1 \cdot \Gamma + K \cdot \Lambda^2 \cdot \Delta) \gamma$ and both matrices on the right are positive semi-definite.

The equality $(K \cdot \Lambda^1 \cdot \Gamma) \vec{\gamma} = 0$ implies that all of $\{(\vec{u}_k \otimes \vec{u}_k) \cdot (\vec{x}_r \otimes \vec{x}_r)\}_{k=1, r=1}^{N-1, P}$ annihilate $\vec{\gamma}$ for each $1 \leq k \leq M$ and $1 \leq r \leq P$, i.e. $0 = \sum_{j=1}^N \bar{u}_j^k \bar{x}_j^r \gamma_j$. Therefore each of the vectors

$\vec{\gamma} \cdot \vec{x}^r = (\bar{x}_1^r \gamma_1, \dots, \bar{x}_N^r \gamma_N)^T$ is orthogonal to each of \vec{u}^k . That is, the vectors $\{\vec{\gamma} \cdot \vec{x}^r\}_1^P$ are contained in the subspace of \mathbb{C}^N perpendicular to the $N-1$ -dimensional subspace of \mathbb{C}^N spanned by $\{\vec{u}^k\}_1^{N-1}$, i.e. a subspace of \mathbb{C}^N of dimension 1. As none of the entries of $\vec{\gamma}$ are 0, the vectors $\{\vec{x}^r\}_1^P$ must all be collinear and the rank of $K \cdot \Lambda^1$ is 1. \square

PROOF OF THEOREM 3.4:

Let

$$A = \frac{\Lambda^2}{\Lambda^1} = \left(\frac{1 - \lambda_i^2 \bar{\lambda}_j^2}{1 - \lambda_i^1 \bar{\lambda}_j^2} \right).$$

Notice that the A is the Pick matrix corresponding to the problem $\{\lambda_i^1 \rightarrow \lambda_i^2\}_1^N$ on \mathbb{D} .

We first show that A is positive semi-definite. Since $f(z_1)$ is the solution to the problem $\{\lambda_i \rightarrow \omega_i\}_1^N$, it is the solution to the one variable problem with data $\{\lambda_i^1 \rightarrow \omega_i\}_1^N$. By Theorem 6.1 the matrix

$$\Gamma_0 = \left(\frac{W_{ij}}{\Lambda_{ij}^1} \right) = \left(\frac{1 - \omega_i \bar{\omega}_j}{1 - \lambda_i^1 \bar{\lambda}_j^1} \right)$$

is positive semi-definite. Furthermore, since the problem is extremal and minimal, Theorem 6.1 implies that $\deg_1(f) = \text{rank}(\Gamma_0) = N-1$. Let K be an active kernel for the original problem. By lemma 6.6 $\text{rank}(K \cdot \Lambda^1) = 1$. Since $(K \cdot \Lambda^1)$ is positive semi-definite with non-zero diagonal entries the fact that it has rank 1 implies that all of its entries are non-zero. The matrix

$$\left(\frac{1}{K \cdot \Lambda^1} \right)_{ij} = \frac{1}{K_{ij} \cdot \Lambda_{ij}^1}$$

is also positive semi-definite. We conclude that A is positive semi-definite by writing $A = \Lambda^2 \cdot \frac{1}{\Lambda^1} = K \cdot \Lambda^2 \cdot \frac{1}{K \cdot \Lambda^1}$ and noticing that the right-hand side is a Schur product of two positive matrices.

To construct V , let $n_1 = \text{rank}(A)$. By the one dimensional Pick theorem there exists a Blaschke product $m(\lambda)$ of degree n_1 such that $m(\lambda_i^1) = \lambda_i^2$. Write m as the ratio of two irreducible polynomials $m(\lambda) = \frac{q(\lambda)}{r(\lambda)}$, let $p(z_1, z_2) = z_2 r(z_1) - q(z_1)$ and notice that $V \cap \mathbb{D}^2 = Z_p \cap \mathbb{D}^2 = \{(\lambda, m(\lambda)) : \lambda \in \mathbb{D}\}$ contains all of the nodes of the original problem. Furthermore, since V is inner and the restriction of f to V has less than N zeros, each solution of the original problem $\{\lambda_i \rightarrow \omega_i\}_1^N$ equals f on V by Theorem 1.7 of [19]. We now examine two cases:

Case i. $\deg_1(f) < \deg_1(p)$. To show that V is a strong Pick set for f , fix a $g \in \mathcal{S}(\mathbb{D}^2)$ that equals f on V . There exists a point $w \in \mathbb{D}$ such that $V \cap \mathbb{D}^2$ contains n_1 distinct

points of the form (l_j, w) . Consider the problem $\{(l_j, w) \rightarrow f(l_j)\}_1^{n_1}$ on \mathbb{D}^2 and consider the matrix $W \cdot K$ associated to this problem with K the Szegő kernel of \mathbb{D}^2 ,

$$W \cdot K = \left(\frac{1 - \overline{f(l_i)}f(l_j)}{(1 - \bar{z}_i z_j)(1 - \bar{w}w)} \right) = \left(\frac{1 - \overline{f(l_i)}f(l_j)}{(1 - \bar{z}_i z_j)(1 - |w|^2)} \right) = \frac{1}{(1 - |w|^2)} \left(\frac{1 - \overline{f(l_i)}f(l_j)}{(1 - \bar{z}_i z_j)} \right).$$

The right most matrix in the above equality has rank equal to $\deg(f) < n_1$ by Theorem 6.1. Thus, $W \cdot K$ is singular, Lemma 6.4 implies that f is the unique solution to the problem $\{(l_j, w) \rightarrow f(l_j)\}_1^{n_1}$ and since g is another solution, $g = f$.

Case ii. $\deg_1(f) \geq \deg_1(p)$. To show that V is not a strong Pick set for f , it will suffice to construct a solution g to the original problem that does not equal f , since in the first part of the proof we showed all solutions agree on V . To construct such a g we modify the argument in Chapter 12 of [4] and show that there exists a pair of positive semi-definite matrices (Γ, Δ) with Δ non-zero such that

$$(6.16) \quad W = \Gamma \cdot \Lambda^1 + \Delta \cdot \Lambda^2.$$

By Lemma 6.5, the existence of such matrices implies that the original problem has a solution that depends on z_2 .

A pair positive semi-definite matrices (Γ, Δ) satisfies 6.16 if and only if

$$(6.17) \quad A \cdot \Delta = \frac{\Lambda^2}{\Lambda^1} \cdot \Delta \leq \Gamma_0$$

in which case $\Gamma = \Gamma_0 - A \cdot \Delta$. Write A as the sum of M rank one matrices $A = \sum_{i=1}^M [x^i \otimes x^i]$

and notice that the rank one matrix $\Delta = \varepsilon[v \otimes v]$ will satisfy (6.17) for some $\varepsilon > 0$ if and only if for each r the vector $v \cdot x^r := (v_1 x_1^r, \dots, v_N x_N^r)^T$ lies in the range of Γ_0 . Since the rank of Γ_0 is $N - 1$, it suffices to fix any non-zero vector u perpendicular to the range of Γ_0 and find v so that for each r the vector $v \cdot x^r$ is perpendicular to u . These two constraints translate into the following system of M linear equations

$$\sum_{i=1}^N v_i x_i^r \bar{u}_i = 0 \text{ for } r = 1, \dots, M.$$

Since $M < N$, there is a non-zero v in \mathbb{C}^N satisfying the above constraints and hence there exists a rank one Δ satisfying $A \cdot \Delta \leq \Gamma_0$. \square

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